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Calculation of the Pontrjagin class for $\mathrm{U}(1)$ instantons on non-commutative $\mathbb{R}^4$

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Abstract: In non-commutative spaces, it is unknown whether the Pontrjagin class gives integer, as well as, the relation between the instanton number and Pontrjagin class is not clear. Here we define "Instanton number" by the size of the matrices B_α in the ADHM construction. We show the analytical derivation of the non-commuatative $\mathrm{U}(1)$ instanton number as an integral of Pontrjagin class (instanton charge) with the Fock space representation. Our approach is for the arbitrary converge non-commutative $\mathrm{U}(1)$ instanton solution, and is based on the anti-self-dual (ASD) equation itself. We give the Stokes' like theorem for the number operator representation. The theorem shows that instanton charge is given by some boundary sum. Using the ASD conditions, we conclude that the instanton charge is equivalent to the instanton number.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

Contents

1. Introduction

Recently, there has been much interest in non-commutative field theory motivated by the string theory [1]–[5]. For example, the non-commutative gauge theory arises on D-branes in the presence of background constant Neveu-Schwarz B-field. The discoveries show us the analysis of non-commutative gauge theory is very important for non-perturbative analysis of the string theory. At the same time non-commutative instantons are one of the great interests for many physicists, since the instanton plays an important role of the non-perturbative analysis of the Yang-Mills theory.

In commutative space, there is a well-known method to construct instanton solution, which is given by Atiyah, Drinfeld, Hitchin and Manin (ADHM) [6, 7, 8]. There is the one-to-one correspondence between the instanton solutions and the ADHM data. On the other hand, in non-commutative spaces case, a pioneering work for instantons was done by Nekrasov and Schwarz [9] (and see also [10]–[18]). They showed that non-commutative instanton solutions are obtained by deformed ADHM equations, where the deformation of the equations is caused by the noncommutativity of spaces. One of the remarkable feature of instantons on non-commutative spaces is their including some kind of resolution of singularities. For example, there is no U(1) instanton in commutative spaces because of small instanton singularities. On the contrary, in non-commutative spaces the point of spacetime is blurred, and the singularities are smeared, therefore instantons exist.

We constructed the elongated type-U(1) instanton solution for arbitrary instanton number [19].[1] Here we defined "instanton number" by the $\dim V$ of the matrices $B_\alpha \in$ Hom (V, V) that appear in ADHM data and V is a complex vector space i.e. $V = \mathbf{C}^k$. In other words, we defined "instanton number" by a rank of projection. In commutative case, instanton number is given by the Pontrjagin class and it is equivalent to $\dim V$. However in non-commutative spaces, many problems are left for the defining of the integer-valued Pontrjagin class, as well as for the proof of the identity between the instanton number and the Pontrjagin class. There are some explanations about the relation between the Pontrjagin class and the instanton number ([10, 13] and so on). They say the integral of the Pontrjagin class is equivalent to the $\dim(1-P)$ and instanton number. (Here P is a projector that appears in the ADHM construction.) As we will see, this result is exactly same as the result of this article. But their explanations are not strict because these discussions contain some gaps and calculations performed including infinity. For example, trace operation has no cyclic symmetry in non-commutative theories (total divergence terms appear when order of operators is changed by the cyclic rotating). When we estimate topological charge like the integral of the Pontrjagin class, the surface terms are essential. So we have to estimate carefully all surface terms including the terms caused by using cyclic rotating in trace operation. This problem is not solved by the pure gauge condition. But there has been no proof that estimate all surface terms. Another type of the problem is infinity. For example, strictly speaking the result $\dim(1-P)$ is not defined because it takes arbitrary value. If we try to estimate them carefully without ambiguity from infinity, there is only way that we introduce some cut-off (or boundary) and calculate all the surface terms. But there has been no proof containing such approach. So we give it in this article. All our calculations are done in finite and all the surface terms are estimated.

In [19] we showed that the elongated type-U(1) instanton solution gives a integer as a Pontrjagin number that is equal to instanton number, by numerical way. We call instanton charge as the integral of the Pontrjagin class in this article to distinguish it from instanton number.

In this article, we perform the calculation of the integral of the Pontrjagin class analytically, for any instanton solution with some converge condition. The elongated type-U(1) instantons belong to the class of instantons that shrink to the origin and its projection operator is diagonalized in the number operator representation consequently. In this case, the instanton number calculus is relatively easy. However, the case that the projection operator is not diagonalized needs some complexity. We note that the cut-off, which we will introduce and finally take infinite limit, plays an important role of the Pontrjagin class calculus.

The organization of this paper is as follows. In section 2, we review the non-commutative instanton and prepare tools of instanton charge calculus. In section 3, instanton charge is calculated in the elongated type instanton case. This calculation is a simple example of

[1] This solution is constructed by the same ADHM data as Braden and Nekrasov used in [20], that is the elongated instanton in "commutative" spaces but with non-trivial metric.

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the general instanton charge calculus. In section 4, the general type of instanton charge is obtained under some converge condition. Stokes' theorem of the Fock space formalism is discussed, too. In section 5, we summarize this article.

2. Non-commutative U(1) instantons

We summarize the non-commutative field theory used in this article and review the non-commutative instantons in this section.

2.1 Non-commutative $\mathbb{R}^4$ and the Fock space representation

Let us consider euclidean non-commutative $\mathbb{R}^4$, whose coordinate functions $x^\mu\,(\mu=1,2,3,4)$ on the deformed non-commutative manifold satisfy the following commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \tag{2.1}$$

where $\theta^{\mu\nu}$ is an antisymmetric real constant matrix, whose elements are called non-commutative parameters. We can always bring $\theta^{\mu\nu}$ to the skew-diagonal form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ -\theta^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & -\theta^{34} & 0 \end{pmatrix} \tag{2.2}$$

by space rotation. For simplicity, we restrict the noncommutativity of the space to the self-dual case of $\theta^{12} = \theta^{34} = -\zeta$ $(\zeta > 0)$. Here we introduce complex coordinates

$$z_1 = \frac{1}{\sqrt{2}}\left(x^1 + ix^2\right), \qquad z_2 = \frac{1}{\sqrt{2}}\left(x^3 + ix^4\right), \tag{2.3}$$

then the commutation relations (2.1) become

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\zeta\,, \qquad \text{others are zero}\,. \tag{2.4}$$

For using the usual operator representation, we define creation and annihilation operators by

$$c_\alpha^\dagger = \frac{z_\alpha}{\sqrt{\zeta}}\,, \qquad c_\alpha = \frac{\bar{z}_\alpha}{\sqrt{\zeta}}\,, \qquad \left[c_\alpha, c_\alpha^\dagger\right] = 1\,, \qquad (\alpha = 1,2)\,. \tag{2.5}$$

The Fock space $\mathcal{H}$ on which the creation and annihilation operators (2.5) act is spanned by the Fock state

$$|n_1, n_2\rangle = \frac{\left(c_1^\dagger\right)^{n_1}\left(c_2^\dagger\right)^{n_2}}{\sqrt{n_1! n_2!}}\,|0,0\rangle\,, \tag{2.6}$$

with

$$\begin{aligned} c_1\,|n_1,n_2\rangle &= \sqrt{n_1}\,|n_1-1,n_2\rangle\,, & c_1^\dagger\,|n_1,n_2\rangle &= \sqrt{n_1+1}\,|n_1+1,n_2\rangle\,, \\ c_2\,|n_1,n_2\rangle &= \sqrt{n_2}\,|n_1,n_2-1\rangle\,, & c_2^\dagger\,|n_1,n_2\rangle &= \sqrt{n_2+1}\,|n_1,n_2+1\rangle\,, \end{aligned} \tag{2.7}$$

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where n_1 and n_2 are the occupation number. The number operators are also defined by

$$\hat{n}_\alpha = c_\alpha^\dagger c_\alpha\,, \qquad \hat{N} = \hat{n}_1 + \hat{n}_2\,, \tag{2.8}$$

which act on the Fock states as

$$\hat{n}_\alpha \left|n_1, n_2\right\rangle = n_\alpha \left|n_1, n_2\right\rangle\,, \qquad \hat{N}\left|n_1, n_2\right\rangle = (n_1 + n_2)\left|n_1, n_2\right\rangle\,. \tag{2.9}$$

In the operator representation, derivatives of a function $f(z_1, \bar{z}_1, z_2, \bar{z}_2)$ are defined by

$$\partial_\alpha f(z) = \left[\hat{\partial}_\alpha, f(z)\right], \qquad \partial_{\bar{\alpha}} f(z) = \left[\hat{\partial}_{\bar{\alpha}}, f(z)\right], \tag{2.10}$$

where $\hat{\partial}_\alpha = \bar{z}_\alpha/\zeta$, $\hat{\partial}_{\bar{\alpha}} = -z_\alpha/\zeta$ which satisfy

$$\left[\hat{\partial}_\alpha, \hat{\partial}_{\bar{\alpha}}\right] = -\frac{1}{\zeta}\,. \tag{2.11}$$

The integral on non-commutative $\mathbb{R}^4$ is defined by the standard trace in the operator representation,

$$\int d^4x = \int d^4z = (2\pi\zeta)^2\, \mathrm{Tr}_{\mathcal{H}}\,. \tag{2.12}$$

Note that $\mathrm{Tr}_{\mathcal{H}}$ represents the trace over the Fock space whereas the trace over the gauge group is denoted by $\mathrm{tr}_{\mathrm{U}(N)}$.

2.2 Non-commutative gauge theory and instantons

Let us consider the $\mathrm{U}(N)$ Yang-Mills theory on non-commutative $\mathbb{R}^4$.

In the non-commutative space, the Yang-Mills connection is defined by

$$\hat{\nabla}_\mu \Psi = -\Psi \hat{\partial}_\mu + \hat{D}_\mu \Psi\,, \tag{2.13}$$

where Ψ is a matter field and D_μ are anti-hermitean gauge fields [11, 21, 22]. Then the Yang-Mills curvature of the connection ∇_μ is

$$F_{\mu\nu} = \left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] = -i\theta_{\mu\nu} + \left[\hat{D}_\mu, \hat{D}_\nu\right]. \tag{2.14}$$

In our notation of the complex coordinates (2.3) and (2.4), the curvature (2.14) is

$$F_{\alpha\bar{\alpha}} = \frac{1}{\zeta} + \left[\hat{D}_\alpha, \hat{D}_{\bar{\alpha}}\right], \qquad F_{\alpha\bar{\beta}} = \left[\hat{D}_\alpha, \hat{D}_{\bar{\beta}}\right], \qquad (\alpha \neq \beta)\,. \tag{2.15}$$

The Yang-Mills action is given by

$$S = -\frac{1}{g^2}\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{tr}_{\mathrm{U}(N)}\, F \wedge *F\,, \tag{2.16}$$

where we denote $\mathrm{tr}_{\mathrm{U}(N)}$ as a trace for the gauge group $\mathrm{U}(N)$, g is the Yang-Mills coupling and $*$ is Hodge-star.

Then the equation of motion is

$$[\nabla_\mu, F_{\mu\nu}] = 0\,. \tag{2.17}$$

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(Anti-)instanton solutions are special solutions of (2.17) which satisfy the (anti-)self-duality ((A)SD) condition

$$F = \pm * F\,. \tag{2.18}$$

These conditions are rewritten in the complex coordinates as

$$F_{1\bar{1}} = +F_{2\bar{2}}\,, \qquad F_{1\bar{2}} = F_{\bar{1}2} = 0 \text{ (self-dual)}\,, \tag{2.19}$$

$$F_{1\bar{1}} = -F_{2\bar{2}}\,, \qquad F_{12} = F_{\bar{1}\bar{2}} = 0 \text{ (anti-self-dual)}\,. \tag{2.20}$$

In the commutative spaces, solutions of eq. (2.18) are classified by the topological charge (integral of the Pontrjagin class)

$$Q = -\frac{1}{8\pi^2}\int \mathrm{tr}_{\mathrm{U}(N)}\, F \wedge F\,, \tag{2.21}$$

which is always integer and called instanton number k. However, in the non-commutative spaces above statement is unclear. We discuss this issue in this article by using the operator representation of (2.21)

$$Q = \begin{cases} \zeta^2\, \mathrm{Tr}_{\mathcal{H}}\ \mathrm{tr}_{\mathrm{U}(N)}(F_{1\bar{1}}F_{2\bar{2}} - F_{12}F_{\bar{1}\bar{2}})\,, & \text{(self-dual)}\,, \\ \zeta^2\, \mathrm{Tr}_{\mathcal{H}}\ \mathrm{tr}_{\mathrm{U}(N)}(F_{1\bar{1}}F_{2\bar{2}} - F_{1\bar{2}}F_{2\bar{1}})\,, & \text{(anti-self-dual)}. \end{cases} \tag{2.22}$$

2.3 Nekrasov-Schwarz non-commutative U(1) instantons

In the ordinary commutative spaces, there is a well-known way to find ASD configurations of the gauge fields. It is ADHM construction which is proposed by Atiyah, Drinfeld, Hitchin and Manin [6]. Nekrasov and Schwarz first extended this method to non-commutative cases [9]. Especially U(1) cases are discussed in [10, 11, 21, 23] in detail. Here we review briefly on the ADHM construction of U(1) instantons [11, 21].

The first step of ADHM construction on non-commutative $\mathbb{R}^4$ is looking for matrices B_1, B_2, I and J which satisfy the deformed ADHM equations

$$\left[B_1, B_1^\dagger\right] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 2\zeta\,, \tag{2.23}$$

$$[B_1, B_2] + IJ = 0\,, \tag{2.24}$$

where B_1 and B_2 are $k \times k$ complex matrices, I and $J^\dagger$ are $k \times 1$ complex matrices. We call this k "instanton number". Note that the right hand side of eq. (2.23) is caused by the noncommutativity of space. In U(1) cases, if $\zeta > 0$, eq. (2.24) and the stability condition allow us to take $J = 0$ [23, 24]. In [11, 21], a projector which project the Hilbert space $\mathcal{H}$ to the subspace of $\mathcal{H}$ is introduced by

$$P = I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger}\, |0,0\rangle\, G^{-1}\, \langle 0,0|\, e^{\sum_\alpha \beta_\alpha c_\alpha} I\,, \tag{2.25}$$

where $B_\alpha = \sqrt{\zeta}\beta_\alpha$ and G is a normalization factor (hermitean matrix)

$$G = \langle 0,0|\, e^{\sum_\alpha \beta_\alpha c_\alpha} II^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger}\, |0,0\rangle\ . \tag{2.26}$$

Let S be a shift operator which obey the following relations:

$$SS^\dagger = 1\,, \qquad S^\dagger S = 1 - P\,. \tag{2.27}$$

Using the shift operator S, the U(1) ASD gauge fields are given as

$$D_\alpha = \sqrt{\frac{1}{\zeta}} S \Lambda^{-1/2} c_\alpha \Lambda^{1/2} S^\dagger \,, \qquad D_{\bar{\alpha}} = -\sqrt{\frac{1}{\zeta}} S \Lambda^{1/2} c_\alpha^\dagger \Lambda^{-1/2} S^\dagger \,, \tag{2.28}$$

where

$$\Lambda = 1 + I^\dagger \hat{\Delta}^{-1} I \,, \qquad \hat{\Delta} = \zeta \sum_\alpha \left(\beta_\alpha - c_\alpha^\dagger\right)\left(\beta_\alpha^\dagger - c_\alpha\right). \tag{2.29}$$

Note that the subspace projected by P is the kernel of $\hat{\Delta}$, and eq. (2.27) implies that Δ^{-1} is well-defined since $S^\dagger$ (or S) removes the kernel.

3. Instanton charge: the elongated U(1) instanton case

In [19], we calculated the instanton charge (integral of the Pontrjagin class) of elongated U(1) instanton numerically. From the results, it is expected that the instanton charge is integer and equal to instanton number as same as commutative case. Therefore, we justify that statement analytically.

3.1 Elongated U(1) instantons on non-commutative $\mathbb{R}^4$

In this subsection, we review the elongated type-U(1) instanton solutions with arbitrary instanton number k in the non-commutative space [19]. After that, we investigate the property of these solutions to calculate the instanton charge. In the following, we set the noncommutativity of space as $\zeta_1 = \zeta_2 = \zeta$.

We set the ADHM data as

$$B_1 = \sum_{i=1}^{k-1} \sqrt{2i\zeta}\, e_i e_{i+1}^\dagger \,, \qquad B_2 = 0 \,, \qquad I = \sqrt{2k\zeta}\, e_k \,, \qquad J = 0, \tag{3.1}$$

where e_i is defined by

$$e_i^\dagger = (\overset{1}{0}, \ldots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \ldots, \overset{k}{0}) \,. \tag{3.2}$$

These matrices (3.1) satisfy the deformed ADHM eqs. (2.23) and (2.24). These ADHM data correspond to the configuration that k instantons are elongated into $z_1 - \bar{z}_1$ direction. Above ADHM data give the expression for the projection P (2.25) as

$$P = I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} \left|0,0\right\rangle G^{-1} \left\langle 0,0\right| e^{\sum_\alpha \beta_\alpha c_\alpha} I = \sum_{n_1=0}^{k-1} \left|n_1, 0\right\rangle \left\langle n_1, 0\right| . \tag{3.3}$$

This P induces the shift operator (2.27):

$$S^\dagger = \sum_{n_1=0}^{\infty} \left|n_1 + k, 0\right\rangle \left\langle n_1, 0\right| + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} \left|n_1, n_2\right\rangle \left\langle n_1, n_2\right| . \tag{3.4}$$

$\Lambda(\hat{n}_1, \hat{n}_2)$ in gauge fields (2.28) is obtained as follows:

$$\Lambda(\hat{n}_1, \hat{n}_2) = \frac{w_k(\hat{n}_1, \hat{n}_2)}{w_k(\hat{n}_1, \hat{n}_2) - 2k w_{k-1}(\hat{n}_1, \hat{n}_2)} \,, \tag{3.5}$$

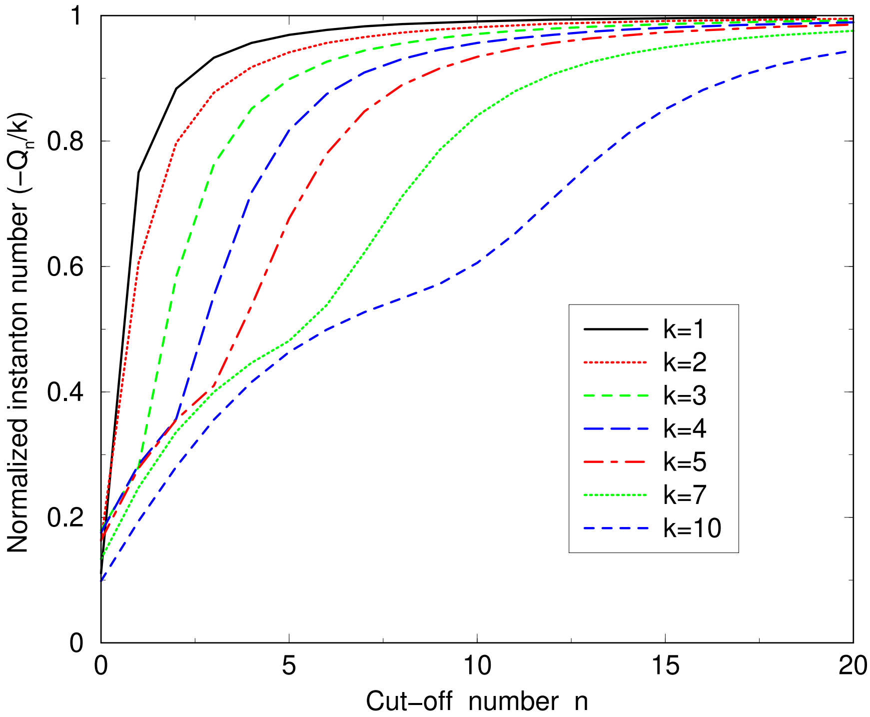

Figure 1: Convergency of the instanton charge [19]; As the cut-off number n increases, all the normalized instanton numbers $-Q_n/k$ approach one. These results show that the instanton charge is equivalent to the instanton number numerically.

where $w_i(n_1, n_2)$ is generated by $F(t)$:

$$w_i(n_1, n_2) = \left(\frac{d}{dt}\right)^i F(t)|_{t=0}, \tag{3.6}$$

$$F(t) = (1-t)^{-n_1+n_2+k-1}(1-2t)^{-n_2-1} = \sum_{i=0}^{\infty} \frac{w_i}{i!} t^i \,. \tag{3.7}$$

In the paper [19], we checked the instanton charge Q of our solution numerically. In its computation, we introduced the Fock space cut-off n as

$$Q = \lim_{n\to\infty} Q_n \,, \tag{3.8}$$

$$Q_n = \zeta^2 \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \langle n_1, n_2| \left(F_{1\bar{1}}F_{2\bar{2}} - F_{1\bar{2}}F_{2\bar{1}}\right) |n_1, n_2\rangle \,. \tag{3.9}$$

The result is shown in figure 1. This result is enough to confirm that the Pontrjagin class of this solution is equivalent to the instanton number in the limit $n \to \infty$.

3.2 Constraints from ASD conditions

We consider constraints on the instanton solution (2.28) from anti-self-dual (ASD) conditions. It is needless to say that the solution (2.28) satisfies the ASD conditions because it is constructed by the ADHM method. However, it is too difficult to get the equations (recursion relations) of Λ (2.29) as explicit forms. This fact make difficulty to analyze the instanton charge analytically. Therefore, we substitute eq. (2.28) for ASD conditions directly and obtain the conditions for Λ, in this section.

The instanton solution (2.28) should satisfy the ASD conditions (2.20):

$$[D_1, D_2] = [D_{\bar{1}}, D_{\bar{2}}] = 0\,, \tag{3.10}$$

$$[D_1, D_{\bar{1}}] + [D_2, D_{\bar{2}}] = -\frac{2}{\zeta}\,. \tag{3.11}$$

We substitute the solution (2.28) for eqs. (3.10) and (3.11), then

$$\begin{aligned}
[D_1, D_2] = {} & \frac{1}{\zeta} S\Big\{\Lambda^{-1/2}(\hat{n}_1,\hat{n}_2)c_1\Lambda^{1/2}(\hat{n}_1,\hat{n}_2)(1-P)\Lambda^{-1/2}(\hat{n}_1,\hat{n}_2)c_2\Lambda^{1/2}\times \\
& \times(\hat{n}_1,\hat{n}_2) - (c_1 \leftrightarrow c_2)\Big\}S^{\dagger} = 0\,,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
[D_1, D_{\bar{1}}] + [D_2, D_{\bar{2}}] = {} & -\frac{1}{\zeta} S\Big\{\Lambda^{-1/2}(\hat{n}_1,\hat{n}_2)c_1\Lambda^{1/2}(\hat{n}_1,\hat{n}_2)(1-P)\Lambda^{1/2}(\hat{n}_1,\hat{n}_2)c_1^{\dagger}\Lambda^{-1/2}\times \\
& \times(\hat{n}_1,\hat{n}_2) - \Lambda^{1/2}(\hat{n}_1,\hat{n}_2)c_1^{\dagger}\Lambda^{-1/2}(\hat{n}_1,\hat{n}_2)(1-P)\Lambda^{-1/2}\times \\
& \times(\hat{n}_1,\hat{n}_2)c_1\Lambda^{1/2}(\hat{n}_1,\hat{n}_2) + (c_1 \leftrightarrow c_2)\Big\}S^{\dagger} = -\frac{2}{\zeta}\,.
\end{aligned} \tag{3.13}$$

$\Lambda^{1/2}(n_1+1, n_2)/\Lambda^{1/2}(n_1, n_2)$ and $\Lambda^{1/2}(n_1, n_2+1)/\Lambda^{1/2}(n_1, n_2)$ appear frequently in the following discussion. Therefore we introduce $h_\alpha(n_1, n_2)$ for convenience:

$$\frac{\Lambda(n_1+1, n_2)}{\Lambda(n_1, n_2)} = 1 - h_1(n_1, n_2)\,, \qquad \frac{\Lambda(n_1, n_2+1)}{\Lambda(n_1, n_2)} = 1 - h_2(n_1, n_2)\,. \tag{3.14}$$

The projector P is not generally diagonalized in the number operator representation, that is, $\langle n_1, n_2|P|n_1',\ n_2'\rangle$ is not proportional to $\delta_{n_1 n_1'}\delta_{n_2 n_2'}$ for arbitrary P. However there is a class that P is diagonalized, which includes elongated U(1) instantons as we saw in eq. (3.3). General case is discussed in section 4, but its calculation is complex. Therefore in this section we treat only diagonal P case as a simple example.

In such cases, eqs.(3.12) and (3.13) are rewritten as

$$\begin{aligned}
& h_1(n_1, n_2+1) - h_1(n_1, n_2) - h_2(n_1+1, n_2) + h_2(n_1, n_2) - h_1(n_1, n_2+1)h_2(n_1, n_2) + \\
& + h_1(n_1, n_2)h_2(n_1+1, n_2) = 0\,,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
& (n_1+1)h_1(n_1, n_2) + (n_2+1)h_2(n_1, n_2) - n_1 h_1(n_1-1, n_2) - \\
& - n_2 h_2(n_1, n_2-1) = 0\,, \qquad ((n_1, n_2) \notin \mathcal{P})\,,
\end{aligned} \tag{3.16}$$

with

$$h_\alpha(n_1, n_2) = 1\,, \qquad ((n_1, n_2) \in \mathcal{P})\,, \tag{3.17}$$

where $\mathcal{P}$ represents the subspace of Fock space which is projected out by P. These recursion relations are used when we estimate the instanton charge in the next subsection.

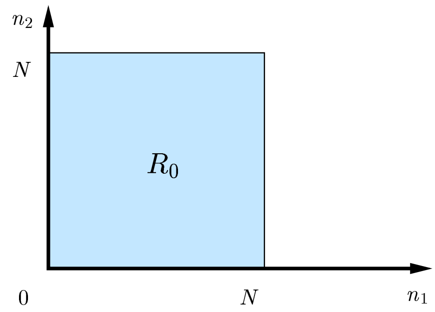

Figure 2: Summation rule (region of $\mathrm{Tr}_{\mathcal{H}}$ operation); This represents the region of summation of $\mathrm{Tr}_{\mathcal{H}}$ operation. We set the cut-off of Fock space in the n_1 and n_2 direction at N.

3.3 Instanton number: elongated instanton case

In this subsection, we calculate analytically the instanton charge of elongated U(1) instantons without the explicit form of the solution and show the charge is equal to the instanton number. Elongated U(1) instantons belong to the class of the instantons whose projection operator P is diagonalized in the number operator representation. This is a good exercise to analyze a general solution in section 4.

The key point is to introduce the cut-off to calculate the instanton charge. We set the cut-off in the n_1 and n_2 directions on N ($N \gg k$) (figure 2), here the "cut-off" means the trace operation cut-off, that is,

$$\mathrm{Tr}_{\mathcal{H}}\big|_{[0,N]}\, O = \sum_{n_1=0}^{N}\sum_{n_2=0}^{N} \langle n_1, n_2|\, O\, |n_1, n_2\rangle\,, \tag{3.18}$$

where O is an arbitrary operator. Note that this cut off is only for the upper bound of summation of the initial state and the final state. To the contrary, upper bound of intermediate summation is frequently over N. To see this phenomena let us consider the trace of eq. (3.11). The trace operation of eq. (3.11) is typically of the form:

$$\mathrm{Tr}_{\mathcal{H}}\big|_{[0,N]} \left(SO'S^\dagger\right) = \sum_{n_1=0}^{N}\sum_{n_2=0}^{N} \langle n_1, n_2|\, SO'S^\dagger\, |n_1, n_2\rangle\,, \tag{3.19}$$

where O' is an arbitrary operator, so summing up the Fock space about O' (sandwiched between S and $S^\dagger$) actually include the outside of cut-off. For example, in the case of elongated instantons (3.4) implies that the above expression is rewritten as

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\big|_{[0,N]} \left(SO'S^\dagger\right) &= \sum_{n_1=k}^{N+k} \langle n_1, 0|\, O'\, |n_1, 0\rangle + \sum_{n_1=0}^{N}\sum_{n_2=1}^{N} \langle n_1, n_2|\, O'\, |n_1, n_2\rangle \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \langle n_1, n_2|\, O'\, |n_1, n_2\rangle\, \theta((n_1, n_2) \in R_s)\,,\end{aligned} \tag{3.20}$$

where $\theta(X)$ is the θ-function whose value is 1 if the proposition X is true and 0 if X is false (figure 3).

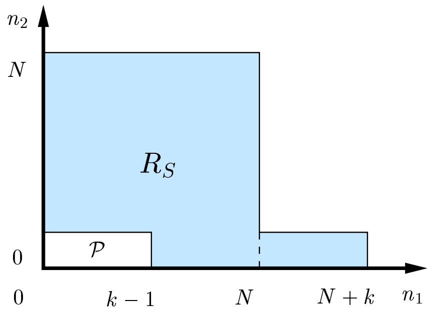

Figure 3: Summation rule (inside of S and $S^\dagger$); This represents the region of intermediate summation of Fock space that is carried out inside of S and $S^\dagger$

For later convenience, we take a trace of ASD condition (3.16):

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}}\big|_{[0,N]}\Big\{2+\zeta[D_1,D_{\bar{1}}]+\zeta[D_2,D_{\bar{2}}]\Big\} = \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big\{(n_1+1)h_1(n_1,n_2)+(n_2+1)h_2(n_1,n_2)- \\
&\qquad - n_1h_1(n_1-1,n_2)-n_2h_2(n_1,n_2-1)\Big\}\theta((n_1,n_2)\in R_s) \\
&= (N+k+1)h_1(N+k,0)-k+(N+1)\times \\
&\quad \times\sum_{n_2=1}^{N}h_1(N,n_2)+(N+1)\sum_{n_1=0}^{N}h_2(n_1,N)-k+\sum_{n_1=N+1}^{N+k}h_2(n_1,0)=0\,. \qquad (3.21)
\end{aligned}$$

Note that in the right hand side of the first equality, only progression of differences appear, so the summation on the cut-off remains. This fact will be generalized as "Stokes' theorem" in the next section. From the above result we obtain the relation

$$\begin{aligned}
E \equiv (N+1)\sum_{n_1=0}^{N}h_2(n_1,N)+(N+1)\sum_{n_2=1}^{N}h_1(N,n_2)+ \\
+\sum_{n_1=N+1}^{N+k}h_2(n_1,0)+(N+k+1)h_1(N+k,0)=2k\,. \qquad (3.22)
\end{aligned}$$

Now let us calculate the instanton charge Q. Q is possible to be written with progression of differences. The detail of the calculation is presented in appendix A. The result is

$$Q_{\mathrm{cutoff}} = k-E+(h_\alpha^2 \text{ terms}) = -k+(h_\alpha^2 \text{ terms})\,, \qquad (3.23)$$

where Q_{cutoff} means

$$\begin{aligned}
Q_{\mathrm{cutoff}} &= \zeta^2\,\mathrm{Tr}_{\mathcal{H}}\big|_{[0,N]}(F_{1\bar{1}}F_{2\bar{2}}-F_{1\bar{2}}F_{2\bar{1}})\,, \qquad (3.24)\\
Q &= \lim_{N\to\infty}Q_{\mathrm{cutoff}}\,. \qquad (3.25)
\end{aligned}$$

Therefore, except for the h_α^2 terms, we obtain the instanton charge $-k$. To identify the charge as the instanton number, we need to take the limit $N\to\infty$. Then we have to know that the behavior of $\hat{\Delta}$ in eq. (2.29) for the Fock state $|n_1,\ n_2\rangle$ in the limit n_1 or $n_2 \to \infty$. By definition eq. (2.29) $\hat{\Delta}$ behaves in this limit as

$$\hat{\Delta} \longrightarrow \zeta(\hat{n}_1 + \hat{n}_2)\,, \qquad (n_\alpha \to \infty)\,. \tag{3.26}$$

From eq. (2.29) thus $\Lambda(\hat{n}_1, \hat{n}_2)$ is obtained as

$$\Lambda(\hat{n}_1, \hat{n}_2) \longrightarrow 1 + \frac{1}{\hat{n}_1 + \hat{n}_2}\,, \qquad (n_\alpha \to \infty)\,, \tag{3.27}$$

which implies

$$\frac{\Lambda(\hat{n}_1+1, \hat{n}_2)}{\Lambda(\hat{n}_1, \hat{n}_2)}\,, \qquad \text{and} \qquad \frac{\Lambda(\hat{n}_1, \hat{n}_2+1)}{\Lambda(\hat{n}_1, \hat{n}_2)} \longrightarrow -\frac{1}{(\hat{n}_1 + \hat{n}_2 + 1)^2}\,, \qquad (n_\alpha \to \infty)\,. \tag{3.28}$$

Therefore, $h_\alpha(\hat{n}_1, \hat{n}_2)$ approaches to

$$h_\alpha(\hat{n}_1, \hat{n}_2) \longrightarrow \frac{1}{(\hat{n}_1 + \hat{n}_2)^2}\,, \qquad (n_\alpha \to \infty)\,. \tag{3.29}$$

As a result, the h_α^2 terms vanish in the limit $N \to \infty$. Finally, we obtain the instanton number:

$$Q = \lim_{N\to\infty} Q_{\text{cutoff}} = -k\,. \tag{3.30}$$

Above calculation is performed only in the elongated U(1) instanton case. However, as long as the projection operator P is diagonalized in the number operator representation, the calculation for other type instantons is same as above. In such case if we put N sufficiently large, then k cells (Fock states) are removed from the $(N+1)\times(N+1)$ square (figure 3) and extra k cells are added outside of the square. Their added k cells' location is arbitrary chosen. However it is easy to show that Q do not depend on the location.

Contrary for the case including the projector P cannot be diagonalized, the calculation is altered. The next section contains to study such cases.

4. Instanton charge: general case

In this section, we construct the instanton charge for general U(1) instanton case. The instanton charge is defined by an integral of the Pontrjagin class (2.22). We find out that it is possible to define the instanton charge under some converge condition, especially the charge looks like conditionally converge. Finally, we see that the instanton charge is equivalent to instanton number defined in ADHM construction, as long as the converge condition is satisfied.

4.1 Instanton equations

In the previous section, the projector P is restricted to the special form of $\sum|n_1, n_2\rangle\,\langle n_1, n_2|$. In that case, instanton charge was defined clearly and calculation with the cut-off was easily done. In this section, we consider the instanton number of an arbitrary instanton solution. The general solution of the instanton is formally given by Nekrasov in [11, 21] as eq. (2.28).

This solution is given formally by ADHM construction. Therefore $\Lambda^{1/2}(\hat{n}_1,\hat{n}_2)$, P and S are possible to be determined by ADHM data formally. However, we do not show concrete expression of them in practice, since the calculation for multi-instantons is too complex. To avoid this problem, though we give the form of instanton solution (2.28) as a result of ADHM construction, we also find constraints condition of $\Lambda^{1/2}(\hat{n}_1,\hat{n}_2)$, P and S not by ADHM data but directly ASD eqs. (3.10) and (3.11).

At first, let us make a more concrete form of S. For example, the eigenvectors of the projector (2.25) are expressed as

$$a^i \equiv I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} \left|0,0\right\rangle e_i \, , \qquad (i = 1 \sim k) \, , \tag{4.1}$$

where e_i is a base of k-dimensional vector. P is a projector onto the k-dimensional subspace and $S^\dagger$ is a map to orthogonal complement of this subspace. We denote the basis of the subspace as a^i. Note that some model might exist such that this expression (4.1) is not valid to make k independent vector. But the following study does not depend on the concrete expression of a^i. We denote the bases of the orthogonal complement as $a_\perp^j (j \geq k+1)$:

$$\left(a_\perp^j\right)^\dagger a^i = 0 \, . \tag{4.2}$$

In the case of (4.1), this condition is written as:

$$\begin{aligned}
\left(a_\perp^j\right)^\dagger a^i &= \overline{\left(a_\perp^j\right)}_{n_1 n_2} \left\langle n_1, n_2\right| I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} \left|0,0\right\rangle e_i \\
&= \overline{\left(a_\perp^j\right)}_{n_1 n_2} \left\langle 0,0\right| \frac{c_1^{n_1}}{\sqrt{n_1!}} \frac{c_2^{n_2}}{\sqrt{n_2!}} I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} \left|0,0\right\rangle e_i \\
&= \overline{\left(a_\perp^j\right)}_{n_1 n_2} I^\dagger \frac{\beta_1^{\dagger n_1}}{\sqrt{n_1!}} \frac{\beta_2^{\dagger n_2}}{\sqrt{n_2!}} e_i = 0 \, ,
\end{aligned} \tag{4.3}$$

where we expand $a_\perp^j$ by the Fock space $\langle n_1, n_2|$ with its coefficient $(a_\perp^j)_{n_1 n_2}$ and $\overline{(a_\perp^j)}_{n_1 n_2}$ is a complex conjugate of $(a_\perp^j)_{n_1 n_2}$. We choose these basis to satisfy the orthonormal conditions,

$$\sum_{n_1,n_2}^{\infty} \overline{(a_\perp^i)}_{n_1 n_2} \left(a_\perp^j\right)_{n_1 n_2} = \delta^{ij} \, . \tag{4.4}$$

The dimension of the $a_\perp^j$ is symbolically $\infty - k$ and $\{a^i\} \bigoplus \{a_\perp^j\}$ is a complete system. For convenience, we rewrite the basis as

$$A^i = \begin{cases} a^i \, , & (i = 1, 2, \dots k) \, , \\ a_\perp^i \, , & (i = k+1, \dots, \infty) \, . \end{cases} \tag{4.5}$$

With this A^i we can express S and $S^\dagger$ concretely as

$$\begin{aligned}
S^\dagger &= \sum_{n,m,n',m',i,j} \overline{A}_{n',m'}^{i+k} S_{ij}^\dagger \left|n', m'\right\rangle \left\langle n, m\right| A_{n,m}^j \, , \\
S &= \sum_{n,m,n',m',i,j} \overline{A}_{n',m'}^{i} S_{ij} \left|n', m'\right\rangle \left\langle n, m\right| A_{n,m}^{j+k} \, .
\end{aligned} \tag{4.6}$$

Since these operators have to obey the relation $SS^\dagger = 1$, the coefficients $A^i_{n,m}$ have to satisfy the normalization condition:

$$\sum_{i,j,p=0}^{\infty} \overline{A}^i_{n,m} S_{ij} S^\dagger_{jp} A^p_{n',m'} = \delta_{nn'} \delta_{mm'} \,. \tag{4.7}$$

Additionally $S^\dagger S = 1 - P$ is expressed as the following form,

$$1 - P = \sum_{\text{all indices}=0}^{\infty} \overline{A}^{i+k}_{n,m} S^\dagger_{ij} A^j_{n',m'} \overline{A}^l_{n',m'} S_{lp} A^{p+k}_{n'',m''} \left|n,m\right\rangle \left\langle n'',m''\right| \tag{4.8}$$

$$= \sum_{\text{all indices}=0}^{\infty} \overline{A}^{i+k}_{n,m} S^\dagger_{ij} S_{jp} A^{p+k}_{n',m'} \left|n,m\right\rangle \left\langle n',m'\right| \,. \tag{4.9}$$

Note that the operators S and $S^\dagger$ are partial isometry operators, but we can choose the matrices S_{ij} and $S^\dagger_{ij}$ as unitary matrices, since the shift of upper indices i of A^i_{nm} in eq. (4.6) is possible to play a role of a shift operator. For convenience, we introduce $K_{n,m,n',m'}$:

$$K_{n,m,n',m'} = \sum \overline{A}^{i+k}_{n,m} S^\dagger_{ij} S_{jp} A^{p+k}_{n',m'} \,. \tag{4.10}$$

Since $(1-P)$ is a projector i.e. $(1-P)^2 = 1-P$, then $K_{n,m,n',m'}$ have to satisfy the following condition:

$$\sum_{n',m'} K_{n,m,n',m'} K_{n',m',l,p} = K_{n,m,l,p} \,. \tag{4.11}$$

The next step, let us make a concrete Λ expression of ASD eqs. (3.12) and (3.13). We substitute eqs. (2.28) and (4.9) for eq. (3.10) as

$$\begin{aligned}
[D_1, D_2] = \frac{1}{\zeta} S \Bigg\{ & \sum K_{n',m',n,m} \frac{\Lambda^{1/2}(n',m')}{\Lambda^{1/2}(n'-1,m')} \frac{\Lambda^{1/2}(n,m+1)}{\Lambda^{1/2}(n,m)} \times \\
& \times \sqrt{n'(m+1)} \left|n'-1,m'\right\rangle \left\langle n,m+1\right| - \\
& - \sum K_{n',m',n,m} \frac{\Lambda^{1/2}(n',m')}{\Lambda^{1/2}(n',m'-1)} \frac{\Lambda^{1/2}(n+1,m)}{\Lambda^{1/2}(n,m)} \times \\
& \times \sqrt{m'(n+1)} \left|n',m'-1\right\rangle \left\langle n+1,m\right| \Bigg\} S^\dagger = 0 \,.
\end{aligned} \tag{4.12}$$

Note that we denote $\hat{n}$ as a number operator and n as a C-number. From eq. (4.12), instanton eq. (3.10) is rewritten as

$$\begin{aligned}
& K_{n'+1,m',n,m-1} \sqrt{(n'+1)m} \, \frac{\Lambda^{1/2}(n'+1,m')}{\Lambda^{1/2}(n,m-1)} = \\
& = K_{n',m'+1,n-1,m} \sqrt{(m'+1)n} \, \frac{\Lambda^{1/2}(n',m'+1)}{\Lambda^{1/2}(n-1,m)} \,.
\end{aligned} \tag{4.13}$$

From eq. (3.11), we get the another equation:

$$
\begin{aligned}
&[D_1, D_{\bar{1}}] + [D_2, D_{\bar{2}}] = \\
&= -\frac{1}{\zeta} S \Bigg\{ K_{n',m',n,m}\sqrt{n'n}\frac{\Lambda^{1/2}(n',m')}{\Lambda^{1/2}(n'-1,m')}\frac{\Lambda^{1/2}(n,m)}{\Lambda^{1/2}(n-1,m)} \left|n'-1,m'\right\rangle \left\langle n-1,m\right| - \\
&\qquad - K_{n',m',n,m}\sqrt{(n'+1)(n+1)} \times \\
&\qquad \times \frac{\Lambda^{1/2}(n'+1,m')}{\Lambda^{1/2}(n',m')}\frac{\Lambda^{1/2}(n+1,m)}{\Lambda^{1/2}(n,m)} \left|n'+1,m'\right\rangle \left\langle n+1,m\right| + \\
&\qquad + K_{n',m',n,m}\sqrt{m'm}\frac{\Lambda^{1/2}(n',m')}{\Lambda^{1/2}(n',m'-1)}\frac{\Lambda^{1/2}(n,m)}{\Lambda^{1/2}(n,m-1)} \left|n',m'-1\right\rangle \left\langle n,m-1\right| - \\
&\qquad - K_{n',m',n,m}\sqrt{(m'+1)(m+1)} \times \\
&\qquad \times \frac{\Lambda^{1/2}(n',m'+1)}{\Lambda^{1/2}(n',m')}\frac{\Lambda^{1/2}(n,m+1)}{\Lambda^{1/2}(n,m)} \left|n',m'+1\right\rangle \left\langle n,m+1\right| \Bigg\} S^\dagger = -\frac{2}{\zeta}\,. \qquad (4.14)
\end{aligned}
$$

From this equation, another recursion relation is obtained as

$$
\begin{aligned}
2K_{l',p',l,p} = {} & K_{l',p',n',m'}K_{n,m,l,p} \times \\
& \times \Bigg\{ K_{n'+1,m',n+1,m}\sqrt{(n'+1)(n+1)}\frac{\Lambda^{1/2}(n'+1,m')}{\Lambda^{1/2}(n',m')}\frac{\Lambda^{1/2}(n+1,m)}{\Lambda^{1/2}(n,m)} - \\
&\quad - K_{n'-1,m',n-1,m}\sqrt{n'n}\frac{\Lambda^{1/2}(n',m')}{\Lambda^{1/2}(n'-1,m')}\frac{\Lambda^{1/2}(n,m)}{\Lambda^{1/2}(n-1,m)} + \\
&\quad + K_{n',m'+1,n,m+1}\sqrt{(m'+1)(m+1)}\frac{\Lambda^{1/2}(n',m'+1)}{\Lambda^{1/2}(n',m')}\frac{\Lambda^{1/2}(n,m+1)}{\Lambda^{1/2}(n,m)} - \\
&\quad - K_{n',m'-1,n,m-1}\sqrt{m'm}\frac{\Lambda^{1/2}(n',m')}{\Lambda^{1/2}(n',m'-1)}\frac{\Lambda^{1/2}(n,m)}{\Lambda^{1/2}(n,m-1)} \Bigg\}. \qquad (4.15)
\end{aligned}
$$

This equation is equivalent to ASD eq. (3.11).

4.2 Stokes' like theorem in number operator representation

In this subsection, we discuss Stokes' like theorem in operator representation. Strictly speaking, this is not equivalent to the Stokes' theorem in usual meaning of commutative space. As far as we know, such theorem had been never studied, but the theorem appears in the calculation of instanton charge and it is very useful.

At first, we consider two-dimensional model:

$$\left[c, c^\dagger\right] = 1\,, \qquad \hat{n} = c^\dagger c\,. \qquad (4.16)$$

Let O be an arbitrary operator, whose Fock state representation is

$$O = \sum_{n,m} O_{n,m} \left|n\right\rangle \left\langle m\right|. \qquad (4.17)$$

The differentiation by $c^\dagger$ is

$$
\begin{aligned}
[c, O] &= \sum_{n,m} O_{n,m} c \left|n\right\rangle \left\langle m\right| - \sum_{n,m} O_{n,m} \left|n\right\rangle \left\langle m\right| c \\
&= \sum_{n,m} O_{n,m}\sqrt{n} \left|n-1\right\rangle \left\langle m\right| - \sum_{n,m} O_{n,m}\sqrt{m+1} \left|n\right\rangle \left\langle m+1\right|. \qquad (4.18)
\end{aligned}
$$

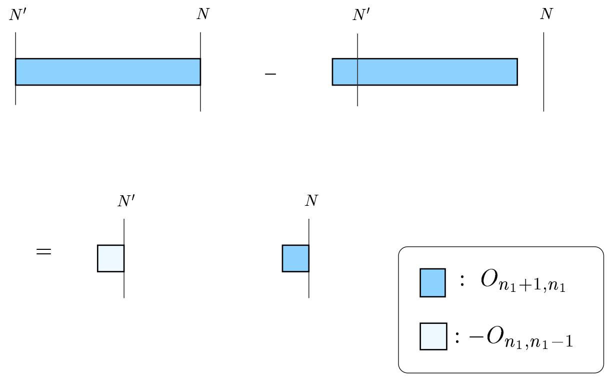

Figure 4: Two-dimensional Stokes' like theorem in the Fock space; integral region is replaced by its boundary.

We define the definite integral of the operator O as the trace of the Fock space:

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\,|_{[N',N]} O &\equiv \sum_{n=N'}^{N} O_{n,n}\,, \\
\lim_{N\to\infty} \mathrm{Tr}_{\mathcal{H}}\,|_{[0,N]} &= \mathrm{Tr}_{\mathcal{H}}\,.
\end{aligned} \tag{4.19}$$

Then the integral of $[c, O]$ is obtained as

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\,|_{[N',N]}[c,O] &= \sum_{n=N'}^{N} \left(O_{n+1,n}\sqrt{n+1} - O_{n,n-1}\sqrt{n}\right) \\
&= O_{N+1,N}\sqrt{N+1} - O_{N',N'-1}\sqrt{N'}\,.
\end{aligned} \tag{4.20}$$

The integral value is determined by the boundary values N and N'. This formula resembles the ordinary Stokes' theorem on the commutative coordinates:

$$\int_D \frac{\partial}{\partial \bar{z}} f(z) dz d\bar{z} = \oint_C f(z) dz\,, \tag{4.21}$$

where D is the integration domain and C is its boundary path. Thus we call the formula (4.20) as the Stokes' like theorem for number operator representation (figure 4). Note that this theorem does not correspond to the ordinary one on the commutative coordinates completely, that is, in the r.h.s. of eq. (4.20) the indices of $O_{N+1,N}$ $(O_{N',N'-1})$ are shifted by one each other. The origin of $O_{N+1,N}$ is cO of differentiation $[c, O]$. $[c, O]$ or O is easily understood by the correspondence to commutative case, but $O_{N+1,N}$ is unclear. So we could not identify the theorem completely with the one on commutative space. There is another problem that we have to study for the theorem. The Stokes' theorem on the commutative space does not depend on the coordinate representation, but in the operator formalism case, we have no idea to check it. It is remained as a problem.

This formula is easily extended to higher dimension. For example, we consider a four-dimensional case. Let O be an arbitrary operator: $O = |n,m\rangle\, O_{nm,n'm'}\, \langle n',m'|$. Then the total derivative of O in the Fock space is $[c_1, O] + [c_2, O]$ and its integration is written as

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\,|_{\mathrm{domain}}([c_1,O]+[c_2,O]) &= \sum_{((n_1,n_2)\in\,\mathrm{domain})} \langle n_1,n_2|\,([c_1,O]+[c_2,O])\,|n_1,n_1\rangle \\
= \sum_{((n_1,n_2)\in\,\mathrm{boundary})} &\Big\{ O_{n_1+1,n_2,n_1,n_2}\sqrt{n_1+1}\,\theta(n_1+1\notin \mathrm{domain}) - \\
&- O_{n_1,n_2,n_1-1,n_2}\sqrt{n_1}\,\theta(n_1-1\notin \mathrm{domain}) + \\
&+ O_{n_1,n_2+1,n_1,n_2}\sqrt{n_2+1}\,\theta(n_2+1\notin \mathrm{domain}) - \\
&- O_{n_1,n_2,n_1,n_2-1}\sqrt{n_2}\,\theta(n_2-1\notin \mathrm{domain})\Big\}\,. \qquad (4.22)
\end{aligned}$$

Here the domain and the boundary are a set of cells belonging some Fock state and a set of its boundary cells (see figure 5 and figure 6). $\theta(n_1+1 \notin \mathrm{domain})$ plays a role of a support on the right side boundary and other θ play similar roles. More higher-dimensional case have a similar expression. On the left side boundary $(n_1 - 1 \notin \mathrm{domain})$ and the down side boundary $(n_2 - 1 \notin \mathrm{domain})$, the summation contribute to the trace with minus sign. This sign is interpreted as orientation of integral. If we assign the orientation to the trace, eq. (4.22) is interpreted as a cyclic integral as same as usual Stokes' theorem.

Next step, let us construct Stokes' like theorem for the covariant derivative with instanton connections

$$D_{\bar\alpha} = -\sqrt{\frac{1}{\zeta}}\, S\Lambda^{1/2}(\hat n_1,\hat n_2)c^\dagger_\alpha \Lambda^{-1/2}(\hat n_1,\hat n_2)S^\dagger\,. \qquad (4.23)$$

Since the creation operator is sandwiched between $S\Lambda^{1/2}(\hat n_1,\hat n_2)$ and $\Lambda^{-1/2}(\hat n_1,\hat n_2)S^\dagger$, $[D_{\bar\alpha}, O]$ is not simple difference form like eq. (4.18). However, when we take a trace (integration) of the commutator, we can find that it is sum of progression of differences:

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}}\,|_{\mathrm{domain}}[D_{\bar 1}, O] = \\
&= \sum^{\infty}_{n,m,n',m',i,j,i',j'} \overline{A}^{i'}_{n',m'} S_{i'j'} S^\dagger_{ij} A^j_{n,m} O_{n,m,n',m'} \times \\
&\times \left(\sum_{l,p} A^{j'+k}_{l+1,p}\frac{\Lambda^{1/2}(l+1,p)}{\Lambda^{1/2}(l,p)}\overline{A}^{i+k}_{l,p}\sqrt{l+1} - A^{j'+k}_{l,p}\frac{\Lambda^{1/2}(l,p)}{\Lambda^{1/2}(l-1,p)}\overline{A}^{i+k}_{l-1,p}\sqrt{l}\right), \qquad (4.24)\\
&\mathrm{Tr}_{\mathcal{H}}\,|_{\mathrm{domain}}[D_{\bar 2}, O] = \\
&= \sum^{\infty}_{n,m,n',m',i,j,i',j'} \overline{A}^{i'}_{n',m'} S_{i'j'} S^\dagger_{ij} A^j_{n,m} O_{n,m,n',m'} \times \\
&\times \left(\sum_{l,p} A^{j'+k}_{l,p+1}\frac{\Lambda^{1/2}(l,p+1)}{\Lambda^{1/2}(l,p)}\overline{A}^{i+k}_{l,p}\sqrt{p+1} - A^{j'+k}_{l,p}\frac{\Lambda^{1/2}(l,p)}{\Lambda^{1/2}(l,p-1)}\overline{A}^{i+k}_{l,p-1}\sqrt{p}\right). \qquad (4.25)
\end{aligned}$$

Here we use eq. (4.7). When we calculate the instanton charge, these formulas play essential roles similar to the Stokes' theorem in the commutative case.

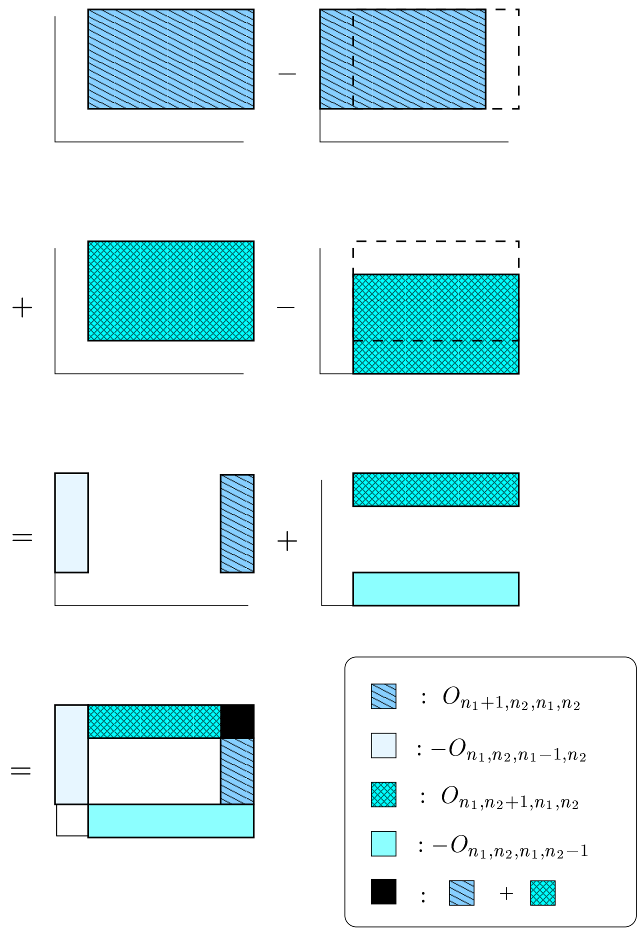

Figure 5: Four-dimensional case; The integral region is a simple rectangle. The summation region is replaced by its boundary.

4.3 Converge condition

In this subsection, we discuss converge conditions for the instanton solution and study the behavior of the solution in the large-$\bar{z}z$ limit.

When the solution is given by (2.25) and the absolute value of the eigenvalue of the β_α is larger than cut-off N, our calculation is not valid. So we do not consider the case that the locations of the instantons are not on the point at infinity. Then we can always chose the cut-off N to be large enough as,

$$0 \le |\beta^e_\alpha| < N\,, \qquad 0 \le |B^e_\alpha| < \sqrt{\zeta}N\,, \qquad \text{for any eigenvalue of } \beta_\alpha(B_\alpha)\,, \tag{4.26}$$

where β^e_α and B^e_α represent eigenvalues of β_α and B_α.

Let us consider the behavior of the gauge connection with the condition (4.26). Since the projector P is given by eq. (2.25), $|P\,|n,m\rangle\,|$ is exponential dumped as n or m increases. On the other hand, $S^\dagger S\,|n,m\rangle = (1-P)\,|n,m\rangle$ approaches $|n,m\rangle$ and this implies

$$\lim_{n\to\infty} K_{n,m,n',m'} = \lim_{m\to\infty} K_{n,m,n',m'} = \delta_{nn'}\delta_{mm'}\,. \tag{4.27}$$

Equation (4.27) is important to the obtain instanton charge. If we claim condition (4.26),

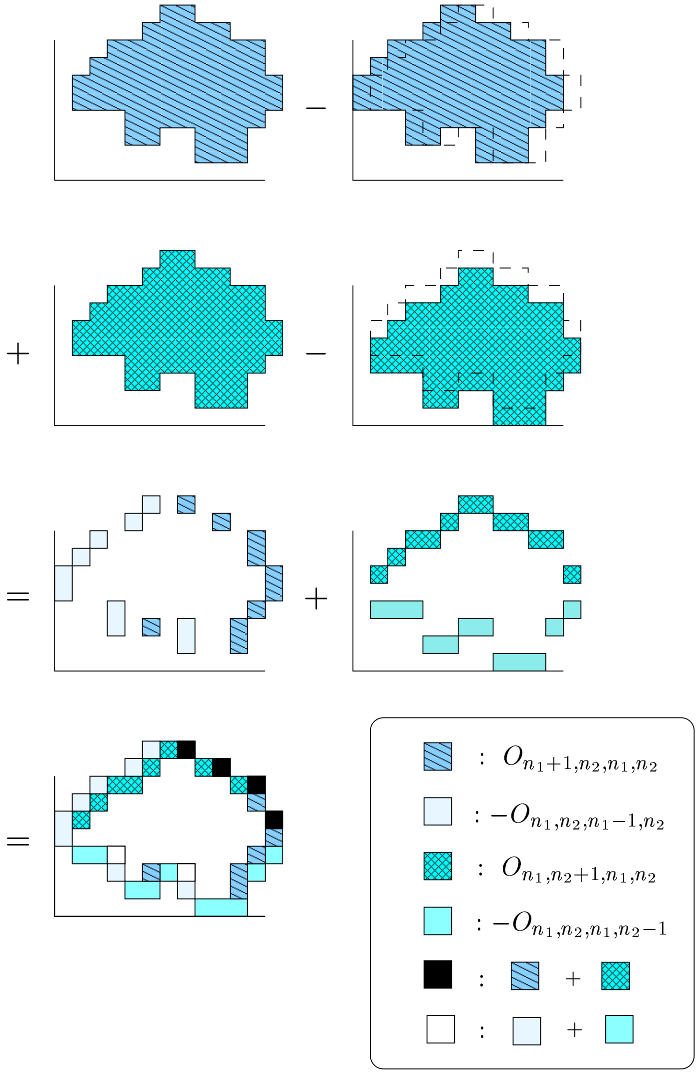

Figure 6: Four-dimensional case; The integral region is an arbitrary case. The summation region is replaced by its boundary.

the difference between $K_{nmn'm}$ and $\delta_{nn'}\delta_{mm'}$ approach 0 as exponential function. But this is too strong condition in practice. To estimate the instanton charge in the next subsection, $K_{nmn'm} - \delta_{nn'}\delta_{mm'} = O(1/(n^2+m^2)^2)$ is a necessary condition. This means that we can define the instanton charge as a converge series for even in the case contains instantons on the point at infinity. Because, if we chose the location of the instantons with the cut-off N whose radius from the origin is divergent in the large-N limit but is enough slower divergent than the cut-off N, then the instanton charge is well defined for the instanton that is on the point of infinity in the large-N limit.

ASD eqs. (4.13) and (4.15) are consistent with these conditions. The $\Lambda(n_1,n_2)$ behavior is given by eq. (3.27) in section 3. Therefore, in the limit that one of n, m, n' and m' approaches ∞, the ASD eqs. (4.13) and (4.15) change to

$$
\begin{aligned}
K_{n'+1,m',n,m-1}\sqrt{(n'+1)m} &= K_{n',m'+1,n-1,m}\sqrt{(m'+1)n}\,, && (4.28)\\
2K_{l',p',l,p} &= K_{l',p',n',m'}K_{n,m,l,p}\,\times\\
&\times\Big\{K_{n'+1,m',n+1,m}\sqrt{(n'+1)(n+1)}-K_{n'-1,m',n-1,m}\sqrt{n'n}\,+\\
&\quad+\,K_{n',m'+1,n,m+1}\sqrt{(m'+1)(m+1)}\,-\\
&\quad-\,K_{n',m'-1,n,m-1}\sqrt{m'm}\Big\}\,, && (4.29)
\end{aligned}
$$

and eq. (4.27) obeys them. These conditions are natural because if number is large enough then the effect from instantons that locate in the are of radius $\sqrt{\zeta}$ do not reach and the gauge field approaches to the trivial connection. As a result of eq. (4.27), we will find soon that calculation of instanton charge become very simple with the Stokes' theorem of the $D_{\bar{\alpha}}$.

The preparations to calculate the instanton charge for general cases are complete. In the next subsection, we start the calculation.

4.4 Instanton number: general case

Let us carry out the calculation of the instanton charge with only primitive methods. The instanton number is written as

$$
\begin{aligned}
Q &= -\,\mathrm{Tr}_{\mathcal{H}}\,1+\frac{\zeta^2}{2}\,\mathrm{Tr}_{\mathcal{H}}\Big\{[D_1,D_{\bar{1}}][D_2,D_{\bar{2}}]+[D_2,D_{\bar{2}}][D_1,D_{\bar{1}}]\,-\\
&\qquad-\,[D_1,D_{\bar{2}}][D_2,D_{\bar{1}}]-[D_2,D_{\bar{1}}][D_1,D_{\bar{2}}]\Big\} && (4.30)\\
&= -\,\mathrm{Tr}_{\mathcal{H}}\,1+\frac{\zeta^2}{2}\,\mathrm{Tr}_{\mathcal{H}}\Big\{[D_{\bar{2}},\ D_2D_{\bar{1}}D_1-D_1D_{\bar{1}}D_2]+[D_{\bar{1}},\ D_1D_{\bar{2}}D_2-D_2D_{\bar{2}}D_1]\Big\}\,,
\end{aligned}
$$

where we use the ASD eqs. (4.12) and (4.14). In the following, we introduce a cut off N like the previous section and the trace is defined with the cut off N.

There are some points that we must make clarify. If we defined the trace of 1 by the $\sum_{n,m=0}^{N}1$, how can we take the sum of intermediate process? As an example, let us consider the trace of $S^\dagger S$. Remind that we choose the orthogonal basis A^i as eq. (4.5). When the cut off is given by N we have to prepare $(N+1)^2+k$ Fock space basis, because the S and $S^\dagger$ are defined by A^i and A^{i+k} for $i=1,\ldots,(N+1)^2$. Using the convergent condition in the previous subsection, orthogonal condition (4.27) is realized for sufficient large N, then $(N+1)^2+k$ Fock states is possible to expand the $(N+1)^2+k$ orthonormal basis A^i. In this situation, the trace of $S^\dagger S$ is written explicitly as

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\,|_{\text{cut off}}S^\dagger S &= \sum_{n,m=0}^{\text{cut off }N}\sum_{i,j,p}\overline{A}^{i+k}_{nm}S^\dagger_{ij}S_{jp}A^{p+k}_{nm}\,,\\
&= \sum_{i,j}^{N}S^\dagger_{ij}S_{ji}\,. && (4.31)
\end{aligned}
$$

It is possible that we take S_{ij} as a unitary matrices. (As we saw at the beginning of this section, though S_{ij} with the cut off N is a unitary matrix $\mathrm{U}(N+1)$, S is still a partial isometry operator.) Consequently, eq. (4.30) become as follows:

$$
\mathrm{Tr}_{\mathcal{H}}\,|_{\text{cut off}}S^\dagger S=\mathrm{Tr}_{\mathcal{H}}\,|_{\text{cut off}}\mathbf{1}\,. \qquad (4.32)
$$

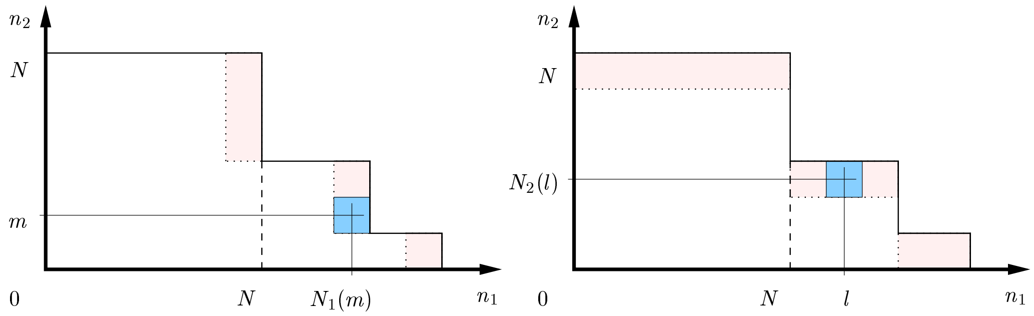

Figure 7: $N_1(m)$, $N_2(l)$; The set of the Fock states is composed of $(N+1)\times(N+1)$ square and extra k cells. The extra k cells a kind of Young-diagram and they are attached to $n_1(n_2)$ axis and the square. $N_1(N_2)$ depend on $n_1(n_2)$.

Someone might think this equation is contradict to $S^\dagger S = 1 - P$. However, infinite series like our instanton charge is sometime conditionally convergent and in such case, we have to introduce a cut off and calculation manner. If we do not provide $(N+1)^2 + k$ Fock states but $(N+1)^2$, then $\sum_{n,m}^{N} \langle n,m| S^\dagger S |n,m\rangle$ is equal to $(N+1)^2 - k$. We can perform the calculation in the both ways, but the later is little complex because we have to estimate the lack of Fock space basis. Therefore, in this study we choose the way to provide $(N+1)^2+k$ basis. For simplicity we take the domain for the sum as (figure 7). It is composite of a $(N+1)\times(N+1)$ square and k cells outside of the square. We can put extra k cells everywhere without inside of the square. However, when the Stokes' like theorem is used, it is convenience to take k cells boundary like (figure 7). As we saw in the previous subsection, Stokes' like theorem change the summation over domain to its on boundary. If we set k cells configuration arbitrary, we have to take account each cells of extra k cells separately into the following calculation. In the following instanton charge calculation, the difference arisen from the configuration of k cells is easily estimated as $O(k/N^2)$. ($O(k/N^2)$ is obtained from h_α linear sum over k points and leading term is not depend on the k cells locations.) We can check this by to perform the following calculation with arbitrary configuration of k cells. Therefore the instanton number do not depend on the locations of these cells in the large-N limit and we perform the calculation with only the domain of (figure 7) for simplicity.

The calculation of instanton number is primitive but sometime complex, so we show the calculation explicitly. For convenience, we introduce $C^{\alpha\bar\beta\beta}$ as

$$D_\alpha D_{\bar\beta} D_\beta = -\sqrt{\frac{1}{\zeta^3}} S C^{\alpha\bar\beta\beta} S^\dagger \,, \tag{4.33}$$

where

$$\begin{aligned} C^{\alpha\bar\beta\beta}_{n,m,n',m'} |n,m\rangle \langle n',m'| = {}& \Lambda^{-1/2}(\hat n_1,\hat n_2) c_\alpha \Lambda^{1/2}(\hat n_1,\hat n_2)(1-P)\Lambda^{1/2}(\hat n_1,\hat n_2) \times \\ & \times\ c^\dagger_\beta \Lambda^{-1/2}(\hat n_1,\hat n_2)(1-P)\Lambda^{-1/2}(\hat n_1,\hat n_2) c_\beta \Lambda^{1/2}(\hat n_1,\hat n_2) \,. \end{aligned} \tag{4.34}$$

The formula (4.25) tells us that the instanton number is given by a sum of progressions of difference and it become the boundary integral. Indeed, explicit expression of

$\mathrm{Tr}_{\mathcal{H}}\,|_{\mathrm{cut\,off}}[D_{\bar{2}}, D_2 D_{\bar{1}} D_1]$ is

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}}\,|_{\mathrm{cut\,off}}[D_{\bar{2}}, D_2 D_{\bar{1}} D_1] = \\
&= -\frac{1}{\zeta^2}\sum_{\text{all indices}}^{\text{cut off } N}\left\{K_{l,p,n,m}K_{n',m',l,p+1}C^{1\bar{2}2}_{n,m,n',m'}\sqrt{p+1}\frac{\Lambda^{1/2}(l,p+1)}{\Lambda^{1/2}(l,p)}-\right.\\
&\qquad\left. -\ K_{l,p-1,n,m}K_{n',m',l,p}C^{1\bar{2}2}_{n,m,n',m'}\sqrt{p}\frac{\Lambda^{1/2}(l,p)}{\Lambda^{1/2}(l,p-1)}\right\} \qquad (4.35)\\
&= -\frac{1}{\zeta^2}\sum_{\text{all indices}}^{\text{cut off } N}\left\{K_{l,N,n,m}K_{n',m',l,N+1}C^{1\bar{2}2}_{n,m,n',m'}\sqrt{N+1}\frac{\Lambda^{1/2}(l,N+1)}{\Lambda^{1/2}(l,N)}\right\}\Bigg|_{N=N_2(l)},
\end{aligned}$$

where the cut off N_2 depend on l as (figure 7).

When we claim the converge condition in previous subsection and we take N large enough, $K_{n,m,n',m'}$ is possible to be substituted by $\delta_{nm}\delta_{n'm'}$ in eq. (4.35). Then the instanton charge is given by the following simple form:

$$\begin{aligned}
Q = &-\sum_{\text{square}} 1 - \sum_n \sqrt{N+1}\times \qquad (4.36)\\
&\times\left\{C^{2\bar{1}1}_{n,N,n,N+1}\frac{\Lambda^{1/2}(n,N+1)}{\Lambda^{1/2}(n,N)} - C^{1\bar{1}2}_{n,N,n,N+1}\frac{\Lambda^{1/2}(n,N+1)}{\Lambda^{1/2}(n,N)}\right\}\Bigg|_{N=N_2(n)} -\\
&-\sum_m \sqrt{N+1}\left\{C^{1\bar{2}2}_{N,m,N+1,m}\frac{\Lambda^{1/2}(N+1,m)}{\Lambda^{1/2}(N,m)} - C^{2\bar{2}1}_{N,m,N+1,m}\frac{\Lambda^{1/2}(N+1,m)}{\Lambda^{1/2}(N,m)}\right\}\Bigg|_{N=N_1(m)},
\end{aligned}$$

where we denote $N = N_\alpha(n)$ as the boundary of the domain (figure 7). Note that the sum of the first term is defined on the only square without extra k cells i.e. it is equal to $(N+1)^2$. But the other sum is defined on the square and the extra k cells because the S and $S^\dagger$ project out k-dimensional subspace from the square. Concrete expression of C for large N is, for example,

$$C^{1\bar{2}2}_{N,m,N+1,m} = \sqrt{(N+1)m^2}\frac{\Lambda^{1/2}(N+1,m)}{\Lambda^{1/2}(N,m)}\frac{\Lambda(N+1,m)}{\Lambda(N+1,m-1)}. \qquad (4.37)$$

Using eq. (4.37) and so on, Q is

$$\begin{aligned}
Q = &-\sum_{\text{square}} 1 - \frac{1}{2}\sum_n\{(N+1)n\,(1-h_1(n-1,N+1))\,(1-h_2(n,N)) -\\
&\qquad -\ (N+1)(n+1)\,(1-h_1(n,N))\,(1-h_2(n,N))\}\,|_{N=N_2(n)} -\\
&-\frac{1}{2}\sum_m\{(N+1)m\,(1-h_2(N+1,m-1))\,(1-h_1(N,m)) -\\
&\qquad -\ (N+1)(m+1)\,(1-h_2(N,m))\,(1-h_1(N,m))\}\,|_{N=N_1(n)}, \qquad (4.38)
\end{aligned}$$

where we use $h_\alpha(n,m)$ defined in eq. (3.14). Since $\Lambda(n+1,m+1)/\Lambda(n,m)$ is written as $(\Lambda(n+1,m)/\Lambda(n,m))(\Lambda(n+1,m+1)/\Lambda(n+1,m))$ or $(\Lambda(n,m+1)/\Lambda(n,m))(\Lambda(n+1,m+1)/\Lambda(n,m+1))$, h_α obeys following consistency condition,

$$h_1(n,m) + h_2(n+1,m) = h_2(n,m) + h_1(n,m+1) + O\left(\frac{1}{(n^2+m^2)^2}\right), \qquad (4.39)$$

where $O(1/(n^2+m^2)^2)$ is the second power of h_α. Using this eq. (4.39), up to $O(1/N)$ the instanton charge Q is written as follows:

$$
\begin{aligned}
Q = &-\sum_{\text{square}} 1 + \frac{1}{2}\sum_n (N+1)|_{N=N_2(n)} + \frac{1}{2}\sum_m (N+1)|_{N=N_1(m)} - \\
&-\sum_n (N+1) h_2(n,N)|_{N=N_2(n)} - \sum_m (N+1) h_1(N,m)|_{N=N_1(m)} - \\
&-\frac{1}{2}\sum_n \{(N+1)\left((n+1)h_1(n,N) - n\, h_1(n-1,N)\right) - \\
&\qquad - \ (N+1)\left((n+1)h_2(n,N) - n\, h_2(n-1,N)\right)\}\,|_{N=N_2(n)} - \\
&-\frac{1}{2}\sum_m \{(N+1)\left((m+1)h_2(N,m) - m\, h_2(N,m-1)\right) - \\
&\qquad - \ (N+1)\left((m+1)h_1(N,m) - m\, h_1(N,m-1)\right)\}\,|_{N=N_1(m)} \,.
\end{aligned}
\tag{4.40}
$$

The last four lines of (4.40) are 0 up to $O(k(N+1)h_\alpha) \sim O(k/N)$.

Taking a trace $\mathrm{Tr}_{[0,N]}$ of eq. (4.14), following equation is obtained:

$$
\begin{aligned}
2\sum_{\text{square}} 1 = &\sum_m (N+1)(1-h_1(N,m))|_{N=N_1(m)} + \\
&+ \sum_n (N+1)(1-h_2(n,N))|_{N=N_2(n)} + O\left(\frac{k}{N}\right).
\end{aligned}
\tag{4.41}
$$

$\sum(N+1)|_{N=N_\alpha}$ is equal to $(N+1)^2+k$ for both $\alpha=1$ and 2. Hence, we finally get the instanton charge as

$$
Q_{\text{cut off}} = -k + O\left(\frac{k}{N}\right),
\tag{4.42}
$$

and it is equivalent to the instanton number in the large-N limit.

5. Conclusion and summary

We have studied the analytical derivation of the instanton number as the integral of the Pontrjagin class (instanton charge) in the Fock space representation. Our approach was for the general non-commutative U(1) instanton solution, and was based on the stability condition and the anti-self dual equation itself. At first the instanton charge of an elongated type was calculated. (In the paper [19], we constructed the non-commutative elongated instanton solution, and performed the instanton charge calculus by the numerical calculation.) After that, general instanton calculation was done by similar manner. To avoid ambiguity, we introduced the cut-off instanton charge and the boundary. All the calculations were done in finite and all the surface terms were estimated. Additionally, all the calculations were performed with algebraic methods. These results shows that the direct relation between the differential geometry and recurrence relations and its series.

Rough sketch of our calculation is following. We made the shift operator with the orthonormal basis. This shift operator played essential role. We gave an orthonormalization which decomposes into the k eigenvectors of projector P and its orthogonal complement. The projector was given by general form, including, e.g. coherent states. We gave the Stokes' like theorem for the number operator representation. The Stokes' like theorem shows that a trace over arbitrary regions of a commutator with coordinate operators is translated into some boundary sum. The commutator with coordinate operators is a derivative, and is a finite-difference form. As same as the commutator with the gauge connection behaves a finite-difference form when we take its trace. For this feature, the trace operation in the instanton charge calculation became the sum over the boundary. Therefore the instanton charge calculus almost came back to elongated type calculation. From this realization we provided the instanton number up to $O(k/N)$, and in large-N limit this difference vanish. Our calculation clarified that the origin of the instanton charge is the dimension of the projection. In other words, the projector in ADHM construction directly reflects the projection in non-commutative field theory.

When we survey our calculation, someone might think the instanton charge is conditionally converge. However it is hasty, because we can define the instanton charge as the sum of each density of each Fock state including constant curvature. If we do not introduce the cut-off then the instanton charge is conditionally convergence in general and its value is meaningless. But we define the instanton charge as $\lim_{N\to\infty} Q_N$ with cut-off N. This Q_N is a converge series since the difference between instanton charge and instanton number is $O(k/N)$ and it is monotone decreasing along increasing N. On the other hand, the cut-off brings about a new problem that we have to check the boundary(cut-off) dependence of the instanton charge. We are able to verify following facts. The instanton charge is not changed when we choose the cut-off not as $N \times N$ square but as $N_1 \times N_2$ rectangle or triangle that is defined by $n_1 + n_2 = N$. In the rectangle boundary case, the difference between instanton charge and instanton number is $O(k/N_1) + O(k/N_2)$ and it is monotone decreasing along both n_1 and n_2 directions. The triangle boundary case is similar too. Further the instanton charge is invariant under the finite deformation of the boundary. Therefore, it is natural to understand that the instanton charge is defined by a converge series whose accumulation value do not depend on the detail of the boundary.

We expect that the extension of this work to a U(N) case is not difficult to do. Furthermore the other type of characteristic class might be calculated as the same manner. These subjects are left for future works.

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A. Calculation of instanton charge: elongated U(1) instantons

In this appendix we show the detail of the calculation of the instanton charge Q in section 3.3.

Q is possible to be written with progression of differences:

$$
\begin{aligned}
Q_{\text{cutoff}} &= \zeta^2 \operatorname{Tr}_{\mathcal{H}}|_{[0,N]} \left\{ F_{1\bar{1}}F_{2\bar{2}} - \frac{1}{2}(F_{1\bar{2}}F_{2\bar{1}} + F_{2\bar{1}}F_{1\bar{2}}) \right\} \\
&= \operatorname{Tr}_{\mathcal{H}}|_{[0,N]} \left\{ -1 + \zeta^2 [D_1, D_{\bar{1}}][D_2, D_{\bar{2}}] - \frac{\zeta^2}{2} \Big([D_1, D_{\bar{2}}][D_2, D_{\bar{1}}] + [D_2, D_{\bar{1}}][D_1, D_{\bar{2}}] \Big) \right\} \\
&= \sum_{(n_1,n_2)} (-1) \times \theta\Big((n_1,n_2) \in R_0\Big) + \sum_{(n_1,n_2)} 1 \times \theta\Big((n_1,n_2) \in R_S\Big) - \\
&\quad -\frac{1}{2} \sum_{(n_1,n_2)} \Bigg[(n_1+1) \Big\{ (n_2+1)(h_1(n_1,n_2) + h_2(n_1,n_2) - h_1(n_1,n_2)h_2(n_1,n_2)) - \\
&\qquad - n_2 \Big(h_1(n_1,n_2-1) + h_2(n_1,n_2-1) - \\
&\qquad\quad - h_1(n_1,n_2-1)h_2(n_1,n_2-1) \Big) \Big\} - \\
&\qquad - n_1 \Big\{ (n_2+1) \Big(h_1(n_1-1,n_2+1) + h_2(n_1,n_2) - \\
&\qquad\quad - h_1(n_1-1,n_2+1)h_2(n_1,n_2) \Big) - \\
&\qquad - n_2 \Big(h_1(n_1-1,n_2) + h_2(n_1,n_2-1) - \\
&\qquad\quad - h_1(n_1-1,n_2)h_2(n_1,n_2-1) \Big) \Big\} + \\
&\qquad + (n_2+1) \Big\{ (n_1+1) \Big(h_1(n_1,n_2) + h_2(n_1,n_2) - h_1(n_1,n_2)h_2(n_1,n_2) \Big) - \\
&\qquad - n_1 \Big(h_1(n_1-1,n_2) + h_2(n_1-1,n_2) - \\
&\qquad\quad - h_1(n_1-1,n_2)h_2(n_1-1,n_2) \Big) \Big\} - \\
&\qquad - n_2 \Big\{ (n_1+1) \Big(h_1(n_1,n_2) + h_2(n_1+1,n_2-1) - \\
&\qquad\quad - h_1(n_1,n_2)h_2(n_1+1,n_2-1) \Big) - \\
&\qquad - n_1 \Big(h_1(n_1-1,n_2) + h_2(n_1,n_2-1) - \\
&\qquad\quad - h_1(n_1-1,n_2)h_2(n_1,n_2-1) \Big) \Big\} \Bigg] \theta\left((n_1,n_2) \in R_S\right).
\end{aligned} \tag{A.1}
$$

The first term and the second term are canceled out. The remaining terms are written by the form of difference, so we obtain the summation only on the cut-off and the result is

$$
\begin{aligned}
Q_{\text{cutoff}} &= k - \frac{1}{2} \Bigg[(N+1) \sum_{n_1=0}^{n} \Big\{ (n_1+1)\left(h_1(n_1,N) + h_2(n_1,N) - h_1(n_1,N)h_2(n_1,N)\right) - \\
&\qquad - n_1 \Big(h_1(n_1-1,N+1) + h_2(n_1,N) - \\
&\qquad\quad - h_1(n_1-1,N+1)h_2(n_1,N) \Big) \Big\} +
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n_1=N+1}^{N+k} \Big\{ (n_1+1)\left(h_1(n_1,0)+h_2(n_1,0)-h_1(n_1,0)h_2(n_1,0)\right) - \\
&\qquad - n_1\left(h_1(n_1-1,0)+h_2(n_1,0)-h_1(n_1-1,0)h_2(n_1,0)\right)\Big\} + \\
&\qquad + (N+k+1)\Big(h_1(N+k,0)+h_2(N+k,0) - \\
&\qquad\qquad - h_1(N+k,0)h_2(N+k,0)\Big) + \\
&+ \sum_{n_2=1}^{N} (N+1)\Big\{ (n_2+1)\left(h_1(N,n_2)+h_2(N,n_2)-h_1(N,n_2)h_2(N,n_2)\right) - \\
&\qquad - n_2\Big(h_1(N,n_2)+h_2(N+1,n_2-1) - \\
&\qquad\qquad - h_1(N,n_2)h_2(N+1,n_2-1)\Big)\Big\}\Bigg]. \qquad \text{(A.2)}
\end{aligned}
$$

Using the relation (3.15), we obtain the form of difference again:

$$
\begin{aligned}
Q_{\text{cutoff}} &= k - \frac{1}{2}\Bigg[(N+1)\sum_{n_1=0}^{N}\Big\{(n_1+1)\left(h_1(n_1,N)-h_2(n_1,N)\right) - \\
&\qquad - n_1\left(h_1(n_1-1,N)-h_2(n_1-1,N)\right)+2h_2(n_1,N)\Big\} + \\
&+ \sum_{n_1=N+1}^{N+k}\Big\{(n_1+1)\left(h_1(n_1,0)-h_2(n_1,0)\right) - \\
&\qquad - n_1\left(h_1(n_1-1,0)-h_2(n_1-1,0)\right)+2h_2(n_1,0)\Big\} + \\
&\qquad + (N+k+1)\left(h_1(N+k,0)+h_2(N+k,9)\right) - \\
&\qquad - (N+1)\sum_{n_2=1}^{N}\Big\{(n_2+1)\left(h_1(N,n_2)-h_2(N,n_2)\right) - \\
&\qquad\qquad - n_2\left(h_1(N,n_2-1)+h_2(N,n_2-1)\right)-2h_1(N,n_2)\Big\} + \\
&+ \frac{1}{2}\Bigg[(N+1)\sum_{n_1=0}^{N}\Big\{\left((n_1+1)h_1(n_1,N)-n_1h_2(n_1-1,N)\right)h_2(n_1,N) + \\
&\qquad + n_1h_1(n_1-1,N+1)h_2(n_1-1,N)\Big\}\Bigg] + \\
&+ (N+1)\sum_{n_2=1}^{N}\Big\{h_1(N,n_2)\left((n_2+1)h_2(N,n_2)-n_2h_2(N,n_2-1)\right) + \\
&\qquad + n_2h_1(N,n_2-1)h_2(N+1,n_2-1)\Big\} + \\
&+ \sum_{n_1=N+1}^{N+k}\Big\{\left((n_1+1)h_1(n_1,0)-n_1h_1(n_1-1,0)\right)h_2(n_1,0) + \\
&\quad + n_1h_1(n_1-1,1)h_2(n_1-1,0)\Big\} + \\
&\quad + (N+k+1)h_1(N+k,0)h_2(N+k,0)\Bigg]. \qquad \text{(A.3)}
\end{aligned}
$$

We perform the summation and get

$$Q_{\text{cutoff}} = k - \Big\{ (N+1)\sum_{n_1=0}^{N} h_2(n_1,N) + (N+1)\sum_{n_2=0}^{N} h_1(N,n_2) + \\ + \sum_{n_1=N+1}^{N+k} h_2(n_1,0) + (N+k+1)h_1(N+k,0)\Big\} + (h_\alpha^2 \text{ terms})\,. \tag{A.4}$$

The relation (3.22) shows that

$$Q_{\text{cutoff}} = k - E + (h_\alpha^2 \text{ terms}) = -k + (h_\alpha^2 \text{ terms})\,. \tag{A.5}$$

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